Lab 7: Butterworth and Chebyshev Filters

Objective

The objective of this lab is to derive the transfer function of Butterworth and Chebyshev filters as well as plot the magnitude response.

1 Butterworth filter

Butterworth filter is a commonly used analogue filter. This filter is characterized by the property that its magnitude response is flat in both passband and stopband. The magnitude-squared response of an N^{th}-order lowpass filter is given by,

$$|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2N}} \tag{1}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_c}\right)^{2N}}} \tag{2}$$

where N is the order of the filter and ω_c is the cutoff frequency in rad/sec.

As the parameter N in Eq. (1) increases, the filter characteristics become sharper: that is, they remain close to unity over more of the passband and become close to zero more rapidly in the stopband, although the magnitude-squared function at the cutoff frequency ω_c will always be equal to one-half because of the nature of Eq. (1).

we can observe the following properties,

- at $\omega = 0$, $|H(j0)|^2 = 1$ for all N
- at $\omega = \omega_c$, $|H(j\omega_c)|^2 = \frac{1}{2}$ for all N, which implies a 3 dB attenuation at ω_c
- $|H(j\omega)|^2$ is a monotonically decreasing function of ω
- $|H(j\omega)|^2$ approaches an ideal lowpass filter as $N \to \infty$
- $|H(j\omega)|^2$ is maximally flat at $\omega = 0$, since derivatives of all orders exist and are equal to zero

The built-in function `[num,den] = butter(N,Wn)` designs an N^{th} order lowpass digital Butterworth filter and returns the filter coefficients in length $N+1$ vectors `num` (numerator) and `den` (denominator). The coefficients are listed in descending powers of z. The cutoff frequency `Wn` must be `0 < Wn < 1`, with 1 corresponding to half the sample rate.

For example, the transfer function of a first-order Butterworth filter with `Wn = 1` rad/sec can be computed as,

```
≫ [num,den] = butter(1,1,'s');
≫ printsys(num,den)
```

The magnitude response of the filter can now be computed as follows,

```
≫ [y,w] = freqs(num,den);
≫ plot(w,abs(y))
```

Where `freqs` function generates frequency response vector `y` and a frequency vector `w`. Later, the magnitude response of the filter is plotted.

1.1 Example

Given that $|H(j\omega)|^2 = \frac{1}{1+64\omega^6}$, determine the analog filter system function $H(s)$.

From the given magnitude-squared response,

$$|H(j\omega)|^2 = \frac{1}{1+64\omega^6} = \frac{1}{1+\left(\frac{\omega}{0.5}\right)^{2(3)}}$$

Comparing this with expression (1), we obtain $N = 3$ and $\omega_c = 0.5$. Observe the gain to decrease by -3 dB at cutoff frequency.

```
clc, clear all, close all
[num,den] = butter(3,0.5,'s');
printsys(num,den)
bode(num,den), grid
```

This results in,

$$H(s) = \frac{0.125}{(s+0.5)(s^2+0.5s+0.25)}$$

$$H(s) = \frac{0.125}{s^3+s^2+0.5s+0.125}$$

To understand better, lets plot the gain of Butterworth low-pass filter of orders 1 through 5, with cutoff frequency $\omega_c = 1$. Note that the slope is $20N$ dB/decade, where N is the filter order.

```
clc, clear all, close all
for i = 1:5
    [num,den] = butter(i,1,'s');
    printsys(num,den)
    bode(num,den), grid, hold on
end
```

Notice the gain becoming a rectangle function as $N \to \infty$.

2 Chebyshev filter

There are two types of Chebyshev filters. The Chebyshev-I filters have a steeper roll-off, equiripple response in the passband and varies monotonically in the stopband. The Chebyshev-II filters have equiripple response in the stopband and is monotonic in the passband. Whereas the Butterworth filters have monotonic response in both bands.

The magnitude-squared response of a Chebyshev-I filter is given by,

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2\left(\frac{\omega}{\omega_c}\right)} \tag{3}$$

where N is the order of the filter, ϵ is the passband ripple factor, which is related to passband ripple R_p in decibels,

$$\epsilon = \sqrt{10^{0.1R_p}-1}$$

$T_N(x)$ is the N^{th}-order Chebyshev polynomial. In the passband, the Chebyshev polynomial alternates between -1 and 1, so the filter gain alternate between maxima at $G=1$ and minima at $G=\frac{1}{\sqrt{1+\epsilon^2}}$. As order N odd (alternatively, even), there are $(N+1)/2$ (alternatively, $N/2$) pass-bank peaks. As ripple parameter ϵ increases, the ripple amplitude and "roll-off" rate increases.

A Chebyshev-II filter is related to the Chebyshev-I filter through a simple transformation. It has a monotone passband and an equiripple stopband, which implies that this filter has both poles and zeros in the s-plane. Therefore the group delay characteristics are better (and the phase response more linear) in the passband than the Chebyshev-I prototype.

If we replace the term $\epsilon^2 T_N^2\left(\frac{\omega}{\omega_c}\right)$ in (3) by its reciprocal and also the argument $x=\frac{\omega}{\omega_c}$ by its reciprocal, we obtain the magnitude-squared response of Chebyshev-II as,

$$|H(j\omega)|^2 = \frac{1}{1+\left[\epsilon^2 T_N^2\left(\frac{\omega_c}{\omega}\right)\right]^{-1}} \tag{4}$$

One approach to designing a Chebyshev-II filter is to design the corresponding Chebyshev-I first and then apply these transformations.

2.1 Example

Lets design a 5^{th}-order analog Butterworth and Chebyshev lowpass filter with a cutoff frequency of 2 GHz. Multiply by 2π to convert the frequency to radians per second. Compute the frequency response of the filters at 4096 points.

The MATLAB script below will plot the attenuation in decibels and shows the frequency in gigahertz.

```
clc, clear all, close all

n = 5; % order
f = 2e9; % cutoff frequency

[zb,pb,kb] = butter(n,2*pi*f,'s');
[bb,ab] = zp2tf(zb,pb,kb);
[hb,wb] = freqs(bb,ab,4096);

[z1,p1,k1] = cheby1(n,3,2*pi*f,'s'); % 3 dB of
    passband ripple
[b1,a1] = zp2tf(z1,p1,k1);
[h1,w1] = freqs(b1,a1,4096);

[z2,p2,k2] = cheby2(n,30,2*pi*f,'s'); % 30 dB
    of stopband attenuation
[b2,a2] = zp2tf(z2,p2,k2);
[h2,w2] = freqs(b2,a2,4096);

plot(wb/(2e9*pi),mag2db(abs(hb)))
hold on
plot(w1/(2e9*pi),mag2db(abs(h1)),'m')
plot(w2/(2e9*pi),mag2db(abs(h2)),'r')
axis([0 4 -40 5])
grid
xlabel('Frequency (GHz)')
ylabel('Attenuation (dB)')
legend('butter','cheby1','cheby2')
```

3 Exercise

1. Use the built-in function `hold on` or otherwise and compute the magnitude response of first order, second-order, third order and fourth order low-pass Butterworth filter with cut-off frequency of 1 radian per second. All plots must appear on the same graph paper. Print your roll # on top of the graph paper.

2. Find poles and zeros of all of the above filters and plot them (separate graph for each filter) onto the s-plane.

3. Use on-line help to plot the magnitude response of a normalized first-order, second-order and third order high-pass Butterworth filter. Repeat for a band-stop Butterworth filter. Choose a range of frequencies (to be stopped) on your own choice.

4. Sketch the magnitude and phase response of a first-order, second order, third-order and fourth order lowpass chebyshev-I filter with passband edge frequency of 10 radians per second and passband ripple of 5 dB. Investigate the impact of changing order on the magnitude response of the system.

5. Draw magnitude response of a third order low-pass chebyshev-I filter when the passband ripple is

0.5dB, 1 dB, 2 dB, 3.5 dB and 5 dB. What is the impact of changing passband ripple on the magnitude response.

6. Sketch magnitude response of a third order band-stop low-pass chebyshev-I filter which stops frequencies from 5 radian per second to 15 radian per second and allow all other frequencies to pass.

7. Plot magnitude response of a fourth order high-pass chebyshev-I filter with passband ripple of 0.5 dB, 2 dB, 3 dB and 5dB which rejects all frequencies below 10 radians per second.

8. Draw magnitude response of a first-order, second order, third order and fourth order low-pass chebyshev-II filter with pass band edge frequency of 10 radians per second and a pass-band ripple of 20 dB.

9. Draw a high-pass chebyshev-II filter which rejects all frequencies below 10 radians per second. Take a ripple = 20 dB.

10. Sketch magnitude response of a third order chebyshev-II band stop filter with passband ripple of 20 dB which stops frequencies from 10 radians per second to 20 radians per second and allows all other frequencies to pass.