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ADVANCED DIGITAL SIGNAL PROCESSING

Lab 6: Discrete Time Fourier Transform (DTFT)

Objective

The objective of this lab is to check and evaluate the frequency response of the system.

1 Introduction

The DTFT of a discrete-time sequence $x(n)$ is a representation of the sequence in terms of the complex exponential sequence $e^{j\omega n}$. If $x(n)$ is absolutely summable, that is, $\sum_{-\infty}^{\infty} |x(n)| < \infty$, then its discrete-time Fourier transform is given by,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{1}$$

Hence, a discrete-time signal $x(n)$ is transformed into a complex-valued continuous function $X(e^{j\omega})$ of real variable ω, called a digital frequency, which is measured in radians/sample.

1.1 Example

Determine the discrete-time Fourier transform of the following finite-duration sequence.

$x(n) = \{ \underset{\uparrow}{1\ 2}\ 3\ 4\ 5 \}$

Using definition in (1),

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \\ &= e^{j\omega} + 2 + 3e^{-j\omega} + 4e^{-j2\omega} + 5e^{-j3\omega} \end{aligned}$$

Since $X(e^{j\omega})$ is a complex-valued function, we will have to plot its magnitude and its angle (or the real and the imaginary part) with respect to ω separately to visually describe $X(e^{j\omega})$. Now ω is a real variable between $-\infty$ and ∞, which would mean that we can plot only a part of the $X(e^{j\omega})$ function using MATLAB. Using two important properties of the discrete-time Fourier transform, we can reduce this domain to the $[0, \pi]$ interval for real-valued sequences.

Lets evaluate $X(e^{j\omega})$ using 501 equispaced points between $[0, \pi]$ and plot its magnitude, angle, real, and imaginary parts.

```
clc, clear all, close all
n = -1:3; x = 1:5; k = 0:500; w = (pi/500)*k; % [0, pi] axis divided into 501 points.
X = x * (exp(-j*pi/500)) .^ (n'*k);
magX = abs(X); angX = angle(X);
realX = real(X); imagX = imag(X);
subplot(2,2,1); plot(w/pi,magX);grid % you can also use k/500 instead of w/pi
xlabel('frequency in \pi units'); title('Magnitude Part')
subplot(2,2,3); plot(w/pi,angX/pi);grid
xlabel('frequency in \pi units'); title('Angle Part')
subplot(2,2,2); plot(w/pi,realX);grid
xlabel('frequency in \pi units'); title('Real Part')
subplot(2,2,4); plot(w/pi,imagX);grid
xlabel('frequency in \pi units'); title('Imaginary Part')
```

Note that we divided the `w` array by `pi` before plotting so that the frequency axes are in the units of π and therefore easier to read.

Note that the angle plot is depicted as a discontinuous function between $-\pi$ and π. This is because the `angle` function in MATLAB computes the principal angle.

2 Frequency Response Function

We plotted magnitude and phase responses in MATLAB by directly implementing their functional forms. MATLAB also provides a function called `freqz` for this computation, which uses the preceding interpretation. In its simplest form, this function is invoked by,

```
>> [H,w] = freqz(num,den,N)
```

which returns the N-point frequency vector `w` and the N-point complex frequency response vector `H` of the system, given its numerator and denominator coefficients in vectors `num` and `den`. The frequency response is evaluated at `N` points equally spaced around the upper half of the unit circle.

The second form,

```
>> [H,w] = freqz(num,den,N,'whole')
```

uses `N` points around the whole unit circle for computation.

In yet another form,

```
>> H = freqz(num,den,w)
```

It returns the frequency response at frequencies designated in vector `w`, normally between `0` and π. It should be noted that the `freqz` function can also be used for numerical computation of the DTFT of a finite-duration, causal sequence $x(n)$. In this approach, `num = x` and `den = 1`.

Given the following transfer function. Plot $|H(e^{j\omega})|$ and $\angle H(e^{j\omega})$.

$$H(z) = \frac{1}{1 - 0.9z^{-1}} \qquad |z| > 0.9$$

To determine the magnitude and phase of $H(e^{j\omega})$. Once again we will use MATLAB to illustrate the use of the `freqz` function. Using its first form, we will take 100 points along the upper half of the unit circle.

```
clc, clear all, close all
num = [1, 0]; % numerator of H(z)
den = [1, -0.9]; % denominator of H(z)
[H,w] = freqz(num,den,100);
magH = abs(H); phaH = angle(H);
subplot(2,1,1);plot(w/pi,magH);grid
xlabel('frequency in \pi units');
ylabel('Magnitude');
title('Magnitude Response')
subplot(2,1,2);plot(w/pi,phaH/pi);grid
xlabel('frequency in \pi units');
ylabel('Phase in \pi units');
title('Phase Response')
```

If you study these plots carefully, you will observe that the plots are computed between $0 \leq \omega \leq 0.99\pi$ and fall short at $\omega = \pi$. This is due to the fact that in MATLAB the lower half of the unit circle begins at $\omega = \pi$. To overcome this problem, we will use the second form of the `freqz` function as follows.

```
>> [H,w] = freqz(num,den,N,'whole')

>> magH = abs(H(1:101));

>> phaH = angle(H(1:101));
```

Now the 101^{st} element of the array `H` will correspond to $\omega = \pi$. A similar result can be obtained using the third form of the `freqz` function.

```
>> w = [0:1:100]*pi/100; H = freqz(num,den,w);

>> magH = abs(H(1:101));

>> phaH = angle(H(1:101));
```

In the future we will use any one of these forms, depending on our convenience. Also note that in the plots we divided the `w` and `phaH` arrays by `pi` so that the plot axes are in the units of π and easier to read.

The frequency response function of a discrete time system is found by substituting $z = e^{j\omega}$ in the pulse transfer function of the system. The following example demonstrates how MATLAB computes and plots the frequency response function.

2.1 Example

Sketch the normalized frequency response function of the system having the pulse transfer function.

$$H(z) = \frac{1 + 0.95z^{-1}}{1 - 1.8z^{-1} + 0.81z^{-2}}$$

Now using the MATLAB script,

```
clc, clear all, close all
k = 256; % frequency points
num = [1 0.95];
den = [1 -1.8 0.81];
w = -pi:pi/(k-1):pi; % frequency vector
h = freqz(num,den,w); % compute frequency
    response of the system
h1 = abs(h); % magnitude response
h2 = h1/(max(h1)); % normalization
h3 = 20*log10(h2); % normalized magnitude shown
     in dB scale
plot(w,h3), xlabel('Frequency in \pi units'),
    ylabel('Magnitude (dB)')
title('Magnitude response')
```

The values of the DTFT of a sequence can be computed if they are described as a rational function in the form of,

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + \cdots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \cdots + d_N e^{-j\omega N}}$$

There are several ways to plot DTFT using MATLAB. Consider the following example.

2.2 Example

Plot the spectrum (DTFT) of the following system.

$$X(e^{j\omega}) = \frac{0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} - 0.033e^{-j3\omega} + 0.008e^{-j4\omega}}{1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} + 1.6e^{-j3\omega} + 0.41e^{-j4\omega}}$$

```
clc, clear all, close all
k = 256; % frequency points
num = [0.008 -0.033 0.05 -0.033 0.008];
den = [1 2.37 2.7 1.6 0.41];
w = 0:pi/(k-1):pi;
h = freqz(num,den,w);
subplot(221)
plot(w/pi,real(h)),grid
title('Real Part')
xlabel('Normalized angular frequency (\omega/\
    pi)')
ylabel('Amplitude')
subplot(222)
plot(w/pi,imag(h)),grid
title('Imaginary Part')
xlabel('Normalized angular frequency (\omega/\
    pi)')
ylabel('Amplitude')
subplot(223)
plot(w/pi,abs(h)),grid
title('Magnitude Spectrum')
xlabel('Normalized angular frequency (\omega/\
    pi)')
ylabel('Magnitude')
subplot(224)
plot(w/pi,angle(h));grid
title('Phase Spectrum')
xlabel('Normalized angular frequency (\omega/\
    pi)')
ylabel('Phase, radians')
```

There is a discontinuity of 2π at $\omega = 0.72$ in the phase response. In such cases, often an alternate type of phase function that is continuous function of ω is derived from the original phase function by removing the discontinuities of 2π. The process of discontinuity removal is called *unwrapping* the phase. In MATLAB, the unwrapping can be implemented using the `unwrap` function.

3 Exercise

1. Sketch the normalized frequency response function of the system having the pulse transfer function.

$$H(z) = \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

   (a) Comment on the response
   (b) Find poles and zeros
   (c) Sketch step, impulse, and ramp response

2. Sketch the magnitude and phase response of the system described by the system function.

$$H(z) = \frac{1}{1 - 0.8z^{-1}}$$

   Note: use the built-in function "angle" for the phase response

3. Put $z = e^{j\omega}$ in example 2.2 and plot the spectrum.

4. Given that,

$$H(z) = \frac{z + 1}{z^2 - 0.9z + 0.81}$$

   is a causal system, find its impulse response representation.